

Avalanches in the Raise and Peel model in the presence of a wall

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We investigate a non-equilibrium one-dimensional model known as the raise and peel model describing a growing surface which grows locally and has non-local desorption. For specific values of adsorption (u_a) and desorption (u_d) rates the model shows interesting features. At $u_a = u_d$, the model is described by a conformal field theory (with conformal charge $c = 0$) and its stationary probability can be mapped to the ground state of the XXZ quantum chain. Moreover, for $u_a \geq u_d$, the model shows a phase in which the avalanche distribution is scale invariant. In this work we study the surface dynamics by looking at avalanche distributions using Finite-size Scaling formalism and explore the effect of adding a wall to the model. The model shows the same universality for the cases with and without a wall for an odd number of tiles removed, but we find a new exponent in the presence of a wall for an even number of avalanches released. We provide new conjecture for the probability distribution of avalanches with a wall obtained by using exact diagonalization of small lattices and Monte-Carlo simulations.

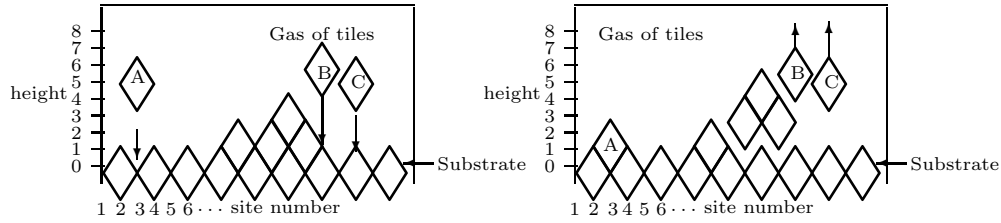


FIG. 1. Three possible cases shown. (A) tile attaches to the surface, (B) tile removes a layer, (C) tile is reflected.

I. INTRODUCTION

The Raise and Peel Model (RPM) is a Markov process first proposed by [1] describing the evolution of a growing surface with a fluctuating interface in one dimension. This model has been found to belong to a new universality class in non-equilibrium phenomena [1–5]. For a particular value of the adsorption (u_a) and desorption (u_d) rates, the model exhibits a phenomenon of self-organized criticality [6, 7] where probability distributions of desorption events show long tails and are characterized by a varying critical exponent that depends on a single parameter given by the ratio of the adsorption and desorption rates [4, 5].

When adsorption and desorption rates are equal, the model becomes solvable. This goes back to a connection established by Razumov and Stroganov [8–10] which relates the two-dimensional dense $O(n = 1)$ fully packed loop models (enumerating the stationary state probability distributions of RPM) to those a ground state wavefunctions of the XXZ chain with L sites [5, 11, 12]. Moreover, the spectra can be obtained by conformal field theory with charge $c=0$ [13, 14]. This offers a nice mathematical structure, which allows to make conjectures using small lattices for expressions of physical quantities that remain valid for quantities for any system size.

In this work, we study the effect on avalanches in the presence of a wall (RPMW) since little is known about this effect when the boundary is allowed to fluctuate. Some other interesting results with the wall have been reported for example in [15, 16]. In Section II we describe the stochastic rules for the model with and without a wall and highlight some of the known results for these two cases. In Section III, we compare the energy spectra of the stochastic Hamiltonian of the XXZ quantum chain for different spin sectors. Lastly in Section IV, we compute critical exponents for avalanche distributions for RPM and RPMW and derive new conjectures for probability expression with a wall.

II. RAISE AND PEEL MODELS

The Raise and Peel Model (RPM) describes a growing and fluctuating interface. An initial configuration is chosen and tiles are dropped onto the surface with a certain probability. Three different processes can happen as shown in figure 1. With some probability $P_i = \frac{1}{L+a-1}$ a tile lands in site $i = 1, \dots, L-1$, while in RPMW the site $i=0$ is chosen with some probability $P_0 = \frac{a}{L+a-1}$. Depending on the slope of surface at the i^{th} site, one of three things can occur.

- Case A: Tile hits a local minimum
if site $i > 0$, with some probability u_a the tile attaches to the substrate, else if site $i=0$ is chosen then half-a-tile attaches to the boundary with rate 1.
- Case B: Tile hits a slope
With Probability u_b the tile peels off tiles within a cluster such as the local local height at every site in the cluster decreases by two: $h_i \rightarrow h_i - 2$ (a tile has a height of 2); In other words, a tile may only remove one layer of tiles above the point of contact.
- Case C: Tile hits a local maxima
Tile reflects and nothing happens.

This stochastic process in continuum time is given by the *master equation* [17, 18]

$$\frac{d}{dt}P_\alpha(t) = - \sum_{\beta} H_{\alpha,\beta} P_\beta(t) \quad (1)$$

where $P_\alpha(t)$ is the (unnormalized) probability of finding the system in one of the states $|\alpha\rangle$ at time t , and $H_{\alpha,\beta}$ is the rate for the transition $|\alpha\rangle \rightarrow |\beta\rangle$. Since this is an intensity matrix, there is at least one zero eigenvalue [4] and its corresponding eigenvector $|0\rangle$ gives the probabilities in the *stationary* state

The functional dependence of the time evolution of distributions can be predicted. The expectation value of observables can be described by stochastic dynamics as:

$$\langle X \rangle(t) = \langle 0 | X e^{-Ht} | \Psi(0) \rangle \quad (17)$$

where the initial state $|\Psi(0)\rangle$ can be expanded in a complete eigenbasis characterizing the system: $|\Psi(0)\rangle = \sum_n c_n |\psi_n\rangle$, and H is the stochastic matrix describing the system. Since H is an intensity matrix, the lowest eigenvalue is zero, hence the lowest non-zero eigenvalue E_1 is expected to dominate the time evolution for large times.

Let us consider the effect of adding a wall on temporal profiles of quantities describing the system. The temporal average of a quantity will be denoted by the Family-Vicsek [28] scaling form:

$$X(t, L) = \frac{x(t, L)}{x(L)} - 1 \sim X\left(\frac{t}{L^z}\right) \quad (18)$$

The average number of clusters $K(x) = \langle \sum_j^L \delta_{h_i, 0} \rangle$ is plotted in Fig. 2 in the form given by Eqn.(18). The data collapse shows that the average number of clusters $K(t, L)$ has a critical exponent given by $z = 1$ for different values of the rate a : ($a=0$ RPM, $a=1$ RPMW). The long time decay is described well by the exponential given by $E_1 = \frac{2\pi v}{L}$ for RPM and by $E_1 = \frac{\pi v}{L}$ for RPMW as is expected from Eqn.(17) for large times.

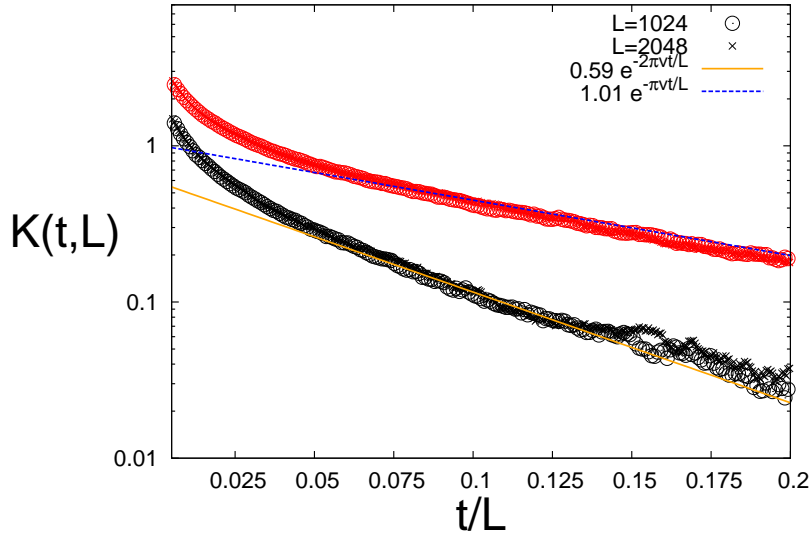


FIG. 2. The time evolution of the average number of clusters $K(t)$ for RPM and RPMW. The lines are the expected decays with $K(t, L) \propto e^{-E_1 t/L}$ with $E_1 = \frac{2v\pi}{L}$ (RPM) and $E_1 = \frac{v\pi}{L}$ (RPMW)

IV. AVALANCHES

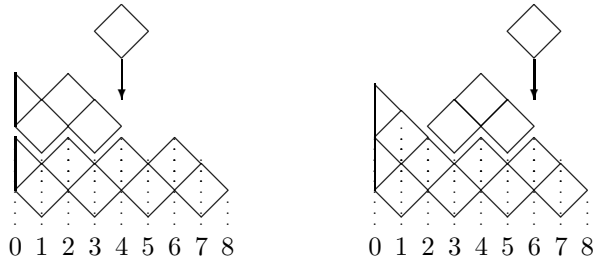


FIG. 3. Example of an even and odd avalanche in RPMW. Avalanches occurring on RPM release only an odd number of tiles

The raise and peel model exhibits events where layers are evaporated from the substrate when a tile from the gas

TABLE II. Estimates for the critical exponents in $S(v, L) \sim v^{-\tau} F(v/L^D)$ for even and odd avalanches using lattices $L=4096$ and $L'=8192$

		1/u	1.0	0.45	0.005
D	RPM [4]	odd	1.004	1.026	1.006
	RPM[this work]	odd	0.992 ± 0.058	1.015 ± 0.006	1.006 ± 0.0001
	RPMW[this work]	odd	0.994 ± 0.025	1.008 ± 0.002	1.006 ± 0.0001
	RPMW[this work]	even	0.980 ± 0.117	1.013 ± 0.008	1.006 ± 0.0001
τ	RPM [4]	odd	3.000	2.25	2.00
	RPM[this work]	odd	2.977 ± 0.079	2.224 ± 0.016	2.011 ± 0.001
	RPMW[this work]	odd	3.003 ± 0.071	2.237 ± 0.012	2.011 ± 0.001
	RPMW[this work]	even	1.932 ± 0.222	1.280 ± 0.074	1.020 ± 0.011

hits the interface. The number of tiles removed defines the size of an avalanche. While this number is always an odd number in RPM, in RPWM there is the possibility of an even number of tiles removed whenever an avalanche touches the boundary. This is illustrated in Figure 3.

It is known that the raise and peel model exhibits self-organized criticality [6, 7] in the regime for $u \geq 1$ [4]. Desorption processes being non-local results in avalanches lacking a characteristic length-scale. Their distribution $S(v, L)$ therefore appears as a power-law which might be described in the finite-size scaling (FSS) form [3, 29]

$$S(v, L) = v^{-\tau} F\left(\frac{v}{L^D}\right) \quad (19)$$

In order to obtain the exponents, the method of moments is used [4, 29]. Using the scaling form (Eqn. 19) we have:

$$\langle v^m \rangle_L = \int S(v, L) v^m dv \quad (20)$$

$$= \int v^{-\tau} F\left(\frac{v}{L^D}\right) v^m dv \quad (21)$$

$$= \int w^{-\tau} L^{-D\tau} F(w) w^m w^{mD} L^D dw \quad (22)$$

$$= L^{D(1+m-\tau)} \underbrace{\int w^{m-\tau} F(w) dw}_{\Gamma_m} \quad (23)$$

$$\Gamma_m \quad (24)$$

where we have used $w \equiv v/L^D$ to get the scaling dependence with L . We can get an estimate for the exponent by looking at the ratio:

$$\langle v^m \rangle_L / \langle v^m \rangle_{L'} = (L/L')^{\sigma(m)} \quad (25)$$

and in this manner the exponent $\sigma(m)$ can be estimated as [1]:

$$\sigma(m) = \frac{\ln(\langle v^m \rangle_L / \langle v^m \rangle_{L'})}{\ln(L/L')} = \begin{cases} 0 & \text{for } m < \tau - 1 \\ D(1 + m - \tau) & \text{for } m > \tau - 1 \end{cases} \quad (26)$$

A linear fit to Eqn.(26) for $m > \tau - 1$ gives an estimate for the values of D and τ . To get an idea of the spread of these values we ran several Monte-Carlo simulations to find the variation of the distribution resulting from different seeds. The results are shown in table II.

Results in this table show that the critical exponents for an *odd* number of tiles removed remains unchanged by adding a wall. However, we found that for an *even* number of tiles the power-law exponent τ decreases by about one. We also found an interesting effect on the finite-size scaling function with the addition of the wall. Fig. 4 shows the scaling function for RPM and RPMW for $u=1$, where we use $\tau = 3.0$ and $\tau = 2.0$ respectively, for *odd* number of tiles and *even* number of tiles, while D is kept fixed at $D = 1$. Fig. 5 shows a similar plot for $1/u=0.005$. The data

collapse for large lattices on these plots confirms the FSS form (Eqn. 19).

V. CONJECTURES FOR PROBABILITIES

Simple conjectures for the probabilities of absorption, desorption, and reflection can be written down by considering the rate of change between the different states. The probability to loose (or gain) v tiles can be written as [1]:

$$P(v, L) = \sum_{\eta \neq \eta'} \delta(v(\eta') - v(\eta) - v) w_{\eta' \rightarrow \eta} P_{\eta'} / \langle 0|0 \rangle \quad (27)$$

where $P_{\eta'} / \langle 0|0 \rangle$ is *normalized* probability to be in the state $|v(\eta')\rangle$ (see Eqn. 10) and $w_{\eta' \rightarrow \eta}$ is the *transition rate* $w_{\eta' \rightarrow \eta}$ from state $|v(\eta')\rangle$ to state $|v(\eta)\rangle$. For $u=1$, the normalization $\langle 0|0 \rangle$ is given by Eqn.(11) and Eqn.(12) for RPM and RPMW, respectively. The probability of absorption (P_a) for a given L has been given by Alcaraz *et al* for RPMW [15]. In this section we will present new conjectures for the probabilities of desorption (P_d) and reflection (P_r) for RPMW. Probabilities for RPM where first reported in [1]. These expressions results in quotients of parabolas as seen in table III. Notice the denominator for the probabilities are different with the addition of the wall since normalization expressions for $\langle 0|0 \rangle$ (Eqns. 11 and 12) are different for the cases with and without a wall [15].

TABLE III. Conjectures for the probabilities of an absorption event $P_a \equiv P(-1, L)$, desorption event $P_d \equiv P(v > 0, L)$, reflection event $P_r \equiv P(0, L)$

Prob.[1, 15]	P_a	P_d	P_r
RPM ($v \in \text{odd}$)	$\frac{3L(L-2)}{4(2L+1)(L-1)}$	$\frac{2(L-2)(L+2)}{4(2L+1)(L-1)}$	—
RPM ($v \in \text{even}$)	—	—	$\frac{3L^2+2L+4}{4(2L+1)(L-1)}$
RPMW ($v \in \text{odd}$)	$\frac{6L^2+8L-5}{4(2L+1)(2L+3)}$	$\frac{4L^2+5L+9}{4(2L+1)(2L+3)}$	—
RPMW ($v \in \text{even}$)	—	$\frac{1.795L-4.562}{4(2L+1)(2L+3)}$	$\frac{6L^2+17.211L+12.50}{4(2L+1)(2L+3)}$

It is interesting to note that although this is a non-equilibrium system, simple expressions for probabilities can be obtained. The mean size of an avalanche $\langle v \rangle$ in the stationary state can be conjectured to be given by a mean-field expression [4].

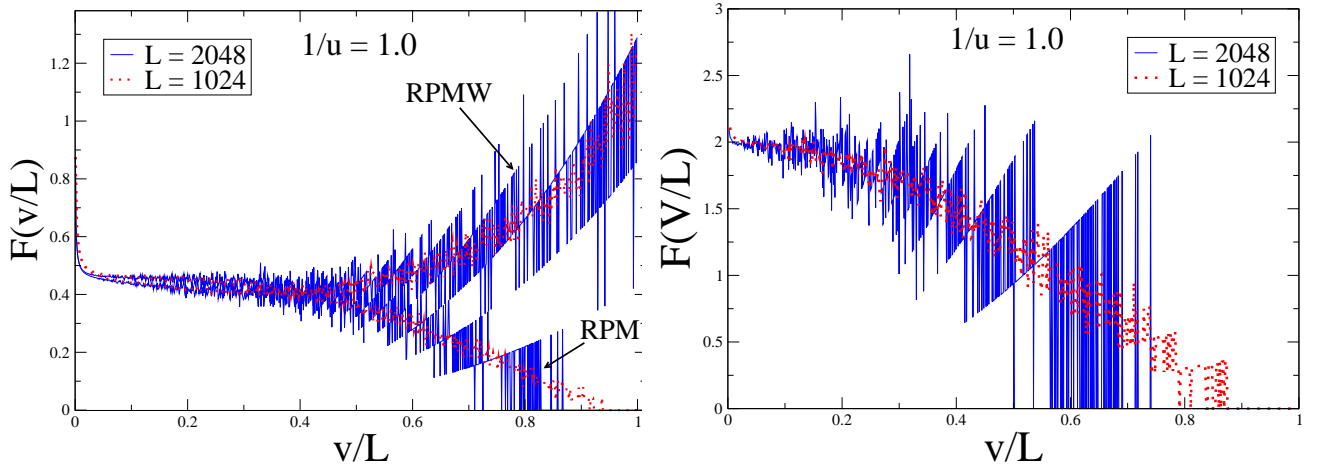


FIG. 4. The scaling function $F(\frac{v}{L})$ with rates $1/u = 1.0$ for $v \in \text{odd}$ number with $\tau = 3.0$ and $D = 1.0$ (LEFT) and $v \in \text{even}$ number with $\tau = 2.0$ and $D = 1.0$

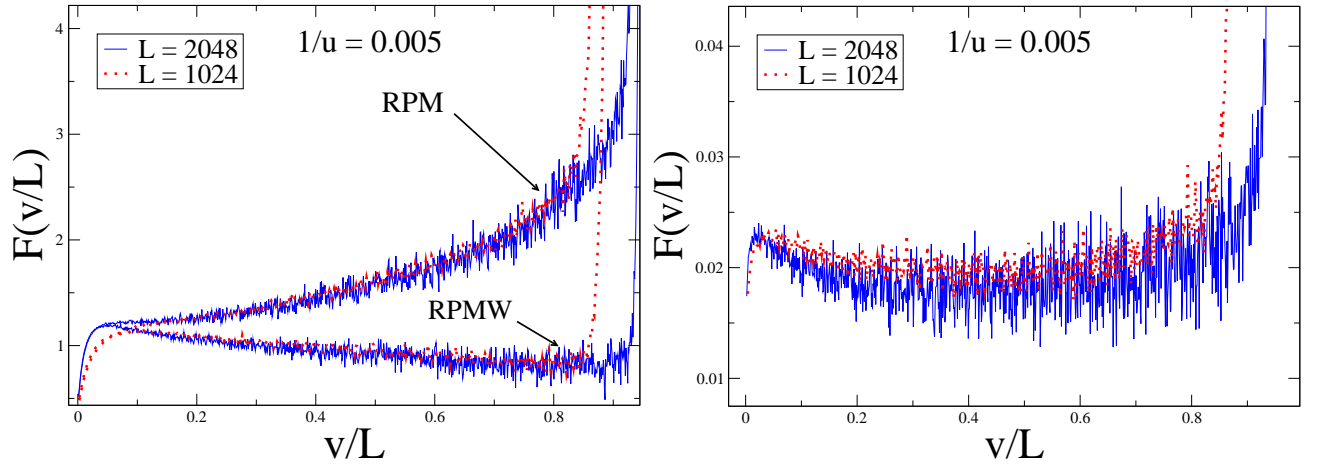


FIG. 5. The scaling function $F(\frac{v}{L})$ with rates $1/u = 0.005$ for $v \in \text{odd}$ number with $\tau = 2.0$ and $D = 1.0$ (LEFT) and $v \in \text{even}$ number with $\tau = 1.0$ and $D = 1.0$

$$\langle v \rangle_L = P_a(L)/P_d(L) \quad (28)$$

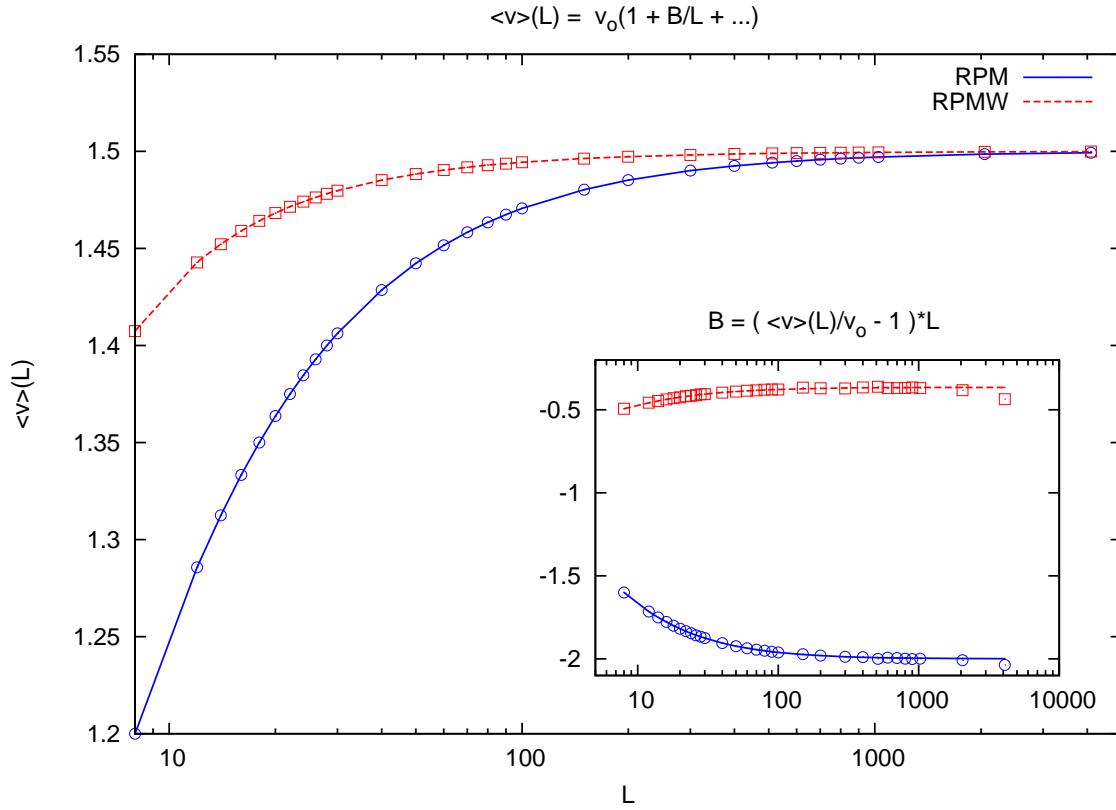


FIG. 6. Average avalanche size $\langle v \rangle(L)$ plotted from the ratio of $P_a(L)/P_d(L)$ (lines) compared to Monte Carlo data for **RPM** (\circ) and **RPMW** (\square)

This is a quotient of parabolas, and in the large limit ($L \gg 1$)

$$\begin{aligned}
\frac{P_a(L)}{P_d(L)} &= \frac{\alpha L^2 + \beta L + \gamma}{aL^2 + bL + c} \\
&= \frac{\alpha L^2 + \beta L + \gamma}{aL^2(1 + \frac{b}{aL} + \frac{c}{aL^2})} \\
&\approx (\frac{\alpha}{a} + \frac{\beta}{aL} + \mathcal{O}(L^{-2}))(1 - \frac{b}{aL} + \mathcal{O}(L^{-2})) \\
&\approx (\frac{\alpha}{a} + \frac{\beta}{aL} - \frac{\alpha b}{a^2 L} + \mathcal{O}(L^{-2})) \\
&= (\frac{\alpha}{a} + \frac{1}{aL}(\beta - \frac{\alpha b}{a})) \\
&= \frac{\alpha}{a}(1 + \frac{1}{L}(\frac{\beta}{\alpha} - \frac{b}{a}))
\end{aligned} \tag{29}$$

We assume that $\alpha \neq 0$ and $a \neq 0$ and dropped terms of the order of $\mathcal{O}(L^{-2})$. From Table III we see that for RPM we have

$$\begin{aligned}
\frac{P_a(L)}{P_d(L)} &= \frac{3}{2}(1 - \frac{1}{L}(\frac{6}{3} - \frac{0}{2})) \\
\Rightarrow \langle v \rangle_L &= \frac{3}{2}(1 - \frac{2}{L})
\end{aligned} \tag{30}$$

while for RPMW we have:

$$\begin{aligned}
\frac{P_a(L)}{P_d(L)} &= \frac{6}{4}(1 + \frac{1}{L}(\frac{8}{6} - \frac{6.795}{4})) \\
\Rightarrow \langle v \rangle_L &= \frac{3}{2}(1 - \frac{0.365}{L})
\end{aligned} \tag{31}$$

The estimated average from the ratio of the conjectures agrees quite well as can be seen in Fig 6. This remarkable since this describes a non-equilibrium system.

It is interesting to see that the leading term on this expansion is universal, whereas the correction term depends on the details of the model (i.e. whether there is a wall or not). This is a similar behavior as in finite-size scaling of the concentration of particles in certain reaction-diffusion systems [30–32] and surface exponent corrections to quantum chains using different types of boundary conditions [33–35].

The simple functional form for the probabilities shown in table III suggests we can guess a general quadratic expression in L for the probabilities $P(v, L)$ by fixing the denominator as:

$$P_{RPM}(v, L) = \frac{a(v)L^2 + b(v)L + c(v)}{4(2L + 1)(L - 1)} \tag{32}$$

$$P_{RPMW}(v, L) = \frac{a(v)L^2 + b(v)L + c(v)}{4(2L + 3)(2L + 1)} \tag{33}$$

and fitting for the parameters $\{a(v), b(v), c(v)\}$ in the forms (32) and (33). The behavior of these parameters with respect to v is shown in Fig 7. As expected, the quadratic term shows a power-law behavior $\propto v^{-3.0}$. The linear and constant terms however do show different behavior, but we were not able to reduce it to an analytical form. Consistency on the fits demands that (see table III)

$$\sum_{v>0} P_{RPM}(v, L) = \frac{2L^2 + 0L - 8}{4(2L + 1)(L - 1)} \tag{34}$$

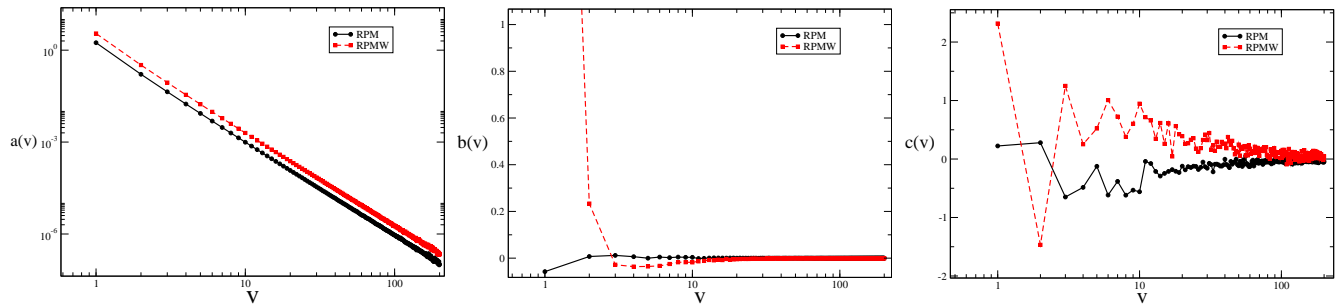


FIG. 7. Behavior of the quadratic term $a(v)$, linear term $b(v)$ and constant term $c(v)$ obtained from fits to Eqn. 32 and Eqn. 33.

$$\sum_{v>0} P_{RPMW}(v, L) = \frac{4L^2 + 5L + 9}{4(2L + 3)(2L + 1)} \quad (35)$$

The sums over the parameters $\sum_v \{a(v), b(v), c(v)\}$ are shown in Table IV for RPM and RPMW. We see that the quadratic $a(v)$ and linear $b(v)$ terms capture the behavior in v quite well, whereas as shown in Fig 7 the constant term $c(v)$ is dominated by fluctuations on the fits.

	$\sum_v a(v)$	$\sum_v b(v)$	$\sum_v c(v)$
RPM	1.99999	-0.00375	-13.8002
RPMW	3.99997	4.96604	30.4445

TABLE IV. Sums over v for the parameters resulting from the fits

VI. CONCLUSION

We studied the non-equilibrium statistical model known as the raise and peel model. We have confirmed that this model retains several features as predicted from conformal invariance for stochastic profiles characterizing the system when changing the boundary conditions. We allowed one boundary to fluctuate and demonstrated that the temporal profile of a stochastic quantity follows its expected behavior from stochastic dynamics where the long time behavior is dominated by the lowest non-zero eigenvalue. We study the surface dynamics by looking at avalanche distributions exhibiting power-law distributions. Using the finite-size scaling formalism we confirm the universality exponent $\tau = 3.0$ for the Raise and Peel with different boundary conditions and we identified an even/odd effect with a new exponent $\tau = 2.0$ for avalanches with an even number of tiles removed. We also found new conjectures for the probability of desorption and reflection with a wall added to the system and checked that they agree with Monte Carlo data.

ACKNOWLEDGEMENTS

We would like to thank M. Henkel and the Groupe de Physique Statistique at Nancy University for helpful discussions. B. W.-K. thankfully acknowledges support from the NSF under the grant PHY-0969689.

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